HYPER VERSION

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Finite baryon density effects on gauge field dynamics

Dietrich Bödeker

Department of Physics and RIKEN BNL Research Center
BNL, Upton, New York 11973, USA
E-mail: bodeker@bnl.gov

Mikko Laine

Theory Division, CERN
CH-1211 Geneva 23, Switzerland
E-mail: mikko.laine@cern.ch

Abstract: We discuss the effective action for QCD gauge fields at finite temperatures and densities, obtained after integrating out the hardest momentum scales from the system. We show that a non-vanishing baryon density induces a charge conjugation (C) odd operator to the gauge field action, proportional to the chemical potential. Even though it is parametrically smaller than the leading C even operator, it could have an important effect on C odd observables. The same operator appears to be produced by classical kinetic theory, allowing in principle for a non-perturbative study of such processes.

Keywords: QCD, Thermal Field Theory, Global Symmetries, Nonperturbative Effects.

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Contents

1. Introduction

The properties of QCD at a finite temperature T and chemical potential μ play an important role for the physics of the Early Universe, heavy ion collision experiments, and neutron stars. Despite the presence of a potentially large scale, say $T \gg \Lambda_{\rm QCD}$, QCD however remains sensitive to non-perturbative infrared physics under these conditions [1, 2]. This means that various effective theories may be derived systematically, but the rich phenomena encoded in them (for a recent review, see [3]) may have to be addressed non-perturbatively.

Quite generically, effective theories tend to possess extra symmetries, only broken by higher dimensional operators. A familiar example is weak interaction induced strangeness violation in zero temperature QCD.

Similar phenomena take place also in high temperature physics. For instance, the dimensionally reduced effective field theory [4] describing the thermodynamics of the electroweak sector of the Standard Model has extra symmetries: parity (P) and charge conjugation (C) are only broken by higher dimensional operators [5].

In QCD, perturbative interactions break neither P nor C, but the latter can be broken by the thermal ensemble, if there is a finite baryon density ($\mu \neq 0$). Indeed, in the dimensionally reduced effective field theory for the thermodynamics of QCD, there is again a new C odd operator, of higher order than the leading C even operator [6].

The purpose of this paper is to address the same phenomenon in the context of the so called Hard Thermal Loop effective theory (for a review and references, see [7]), the generalisation of the dimensionally reduced theory for real time observables. We show that a purely gluonic C odd operator is induced by quark loops. Thus C odd observables may be directly sensitive to non-perturbative bosonic dynamics.

The environment in which we may envisage our results to have significance is mainly that of heavy ion collision experiments. In cosmology the baryon chemical potential is too small to have any significance, $\mu/T \sim 10^{-8}$. In neutron stars, on the other hand, the interesting part of non-perturbative QCD dynamics appears to be related more to quark pairing near the Fermi surface than to gluons [3].

As far as heavy ion collisions go, precision studies of dimensional reduction show that effective theories of the type considered may be quantitatively accurate down to $T \sim 2T_c$ [6], [8]–[11]. At the same time, the infrared dynamics described by the effective theories is completely non-perturbative and does not allow for a perturbative treatment at any reasonable temperature [1, 2, 6, 8, 10, 12, 13, 14]. (Apart perhaps from observables such as the free energy density where non-perturbative effects happen to almost cancel out numerically [15].) Direct four-dimensional (4d) lattice simulations are not available either for real time quantities (for a review, see [16]). Thus effective theories appear presently to be the only way of studying quantitatively some interesting non-perturbative processes for phenomenologically relevant temperatures.

Among the C odd processes one could think of are that various correlation lengths, decay rates and oscillation frequencies change, because previously distinct quantum number channels couple to each other in the presence of $\mu \neq 0$ [17]. In particular the fluctuations of baryon and energy densities are correlated. One could also consider direct C odd observables, such as $M = \sum_{\pi^\pm}(\mathbf{p}_+^2 - \mathbf{p}_-^2)/(\mathbf{p}_+^2 + \mathbf{p}_-^2)$ [18, 19], where the sum is over the momenta of all charged pions. In addition one may consider contributions from C odd amplitudes to C even observables such as dilepton production [20]. We do not here study any of these applications, though.

The plan of the paper is the following. We fix our basic notation and conventions in section 2, review briefly the effective theory in the static limit in section 3, and present our derivation of the general non-static case in section 4. In section 5 we argue that the result of section 4 is equivalent to another effective description, "classical kinetic theory", which also allows for a simple numerical lattice implementation. We conclude in section 6.

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2. Conventions

The partition function of QCD at a finite temperature T and chemical potential μ is

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu Q)} = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\Big(-\frac{1}{\hbar} S_E\Big)\,, \tag{2.1}$$

$$S_E \equiv \int_0^{\beta\hbar} d\tau \int d^3x\, \mathcal{L}_E\,, \tag{2.2}$$

$$\mathcal{L}_E = \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} + \bar{\psi}[\gamma_\mu D_\mu - \gamma_0 \mu]\psi\,, \tag{2.3}$$

where $\beta = T^{-1}$, H is the hamiltonian, Q is the quark number operator (three times the baryon number operator), boundary conditions over the time direction are periodic for A_μ and antiperiodic for $\psi, \bar{\psi}$, $A_\mu = A_\mu^a T^a$, $\mathrm{Tr}\, T^a T^b = \delta^{ab}/2$, $D_\mu = \partial_\mu - igA_\mu$, $F_{\mu\nu} = (i/g)[D_\mu, D_\nu]$, and the metric is $(++++)$. We set $\hbar = 1$ in the following. As we see from eq. (2.3), the chemical potential corresponds in momentum space to shifting fermionic Matsubara frequencies ω_n^f as $\omega_n^f \to \omega_n^f + i\mu$.

The observables we are fundamentally interested in are real time correlation functions of the type

$$C(t_i, x_i; T, \mu) = Z^{-1}\, \mathrm{Tr}\, e^{-\beta(H-\mu Q)} \Big[O_1(t_1,x_1) O_2(t_2,x_2)\ldots\Big], \tag{2.4}$$

where O_i are operators in the Heisenberg picture. As usual, such expectation values can be found by computing first the corresponding objects in the euclidean theory of eqs. (2.1)–(2.3), and performing then an appropriate analytic continuation.

In view of the analytic continuation, we will at a number of points already write the action in Minkowski notation corresponding to the metric $g^{\mu\nu} = \mathrm{diag}\,(+---)$. The continuation will be understood to be made by writing $\tau = it, \partial_\tau = -i\partial_t$, $\omega_n = -i\omega$, $A_0^E = -iA_0^M$. An additional minus sign is inserted in the relation $\mathcal{L}_E(\tau = it) = -\mathcal{L}_M$, such that

$$-S_E = -\int_0^{\beta\hbar} d\tau d^3x\, \mathcal{L}_E \to i \int dt d^3x \mathcal{L}_M \equiv iS_M\,. \tag{2.5}$$

In the continuation scalar products change as

$$a_E \cdot b_E = a_\mu b_\mu \to -a_M \cdot b_M = -a_\mu b^\mu\,, \tag{2.6}$$

where we use the implicit notation that both indices down implies euclidean metric.

We denote the various integrations arising as follows:

$$\int_x = \int d^3x\, dt\,;$$

$$\int_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3}\,;$$

$$
\int_P = \int_{\mathbf{p}} \int \frac{dp_0}{(2\pi)}, \quad P_\mu = (p_0, \mathbf{p})\,;
$$

$$
\sum\!\!\!\!\!\!\!\int_P = \int_{\mathbf{p}} T \sum_{\omega_n}, \quad P_\mu = (\omega_n, \mathbf{p})\,, \tag{2.7}
$$

where ω_n are fermionic (ω_n^f) or bosonic (ω_n^b) Matsubara frequencies; and

$$
\int_v = \int \frac{d\Omega_v}{4\pi}, \qquad v^\mu = (1, v^i)\,, \qquad v_\mu v^\mu = 0\,, \tag{2.8}
$$

where the integral is over the directions Ω_v of v^i. The corresponding δ-functions are assumed normalized such that $\int_x \delta_x = 1$. We also denote

$$
\delta_+(P^2) \equiv 2\theta(p_0)\,(2\pi)\,\delta(P^2)\,, \qquad P^2 \equiv P \cdot P\,, \tag{2.9}
$$

such that

$$
\int_P \delta_+(P^2)\, f(p_0, \mathbf{p}) = \int_{\mathbf{p}} \frac{f(\omega_P, \mathbf{p})}{\omega_P}, \quad \omega_P \equiv |\mathbf{p}|\,. \tag{2.10}
$$

All our results will factorise in a form where a phase space integral is left over, which can be carried out explicitly. Let

$$
n_{\mathrm{F}}(\omega_P) = \frac{1}{e^{\beta\omega_P} + 1}, \qquad N_\pm(\omega_P) = n_{\mathrm{F}}(\omega_P - \mu) \pm n_{\mathrm{F}}(\omega_P + \mu)\,. \tag{2.11}
$$

Then, irrespective of the relative magnitudes of T, μ,

$$
\int_{\mathbf{p}} N_-(\omega_P) = \frac{\mu}{6}\Big(T^2 + \frac{\mu^2}{\pi^2}\Big)\,, \tag{2.12}
$$

$$
\int_{\mathbf{p}} \frac{N_+(\omega_P)}{\omega_P} = -\frac{1}{2}\int_{\mathbf{p}} \frac{\partial N_+(\omega_P)}{\partial \omega_P} = \frac{1}{2}\frac{\partial}{\partial\mu}\int_{\mathbf{p}} N_-(\omega_P) = \frac{1}{4}\Big(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\Big)\,, \tag{2.13}
$$

$$
\begin{aligned}
\int_{\mathbf{p}} \frac{N_-(\omega_P)}{\omega_P^2} &= -\int_{\mathbf{p}} \frac{1}{\omega_P}\frac{\partial N_-(\omega_P)}{\partial \omega_P} = \frac{1}{2}\int_{\mathbf{p}} \frac{\partial^2 N_-(\omega_P)}{\partial \omega_P^2} = \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\int_{\mathbf{p}} N_-(\omega_P) \\
&= \frac{\mu}{2\pi^2}\,.
\end{aligned} \tag{2.14}
$$

3. Effective action at the static limit

We start the actual computation by recalling the fermion induced C odd operators in the static limit. This case corresponds to dimensional reduction [4].

In Minkowski notation, charge conjugation can be defined for the bosonic fields as

$$
A_\mu \to -A_\mu^*\,, \qquad D_\mu \to D_\mu^*\,, \qquad F_{\mu\nu} \to -F_{\mu\nu}^*\,. \tag{3.1}
$$

Furthermore the Minkowski action S_M should be real. Thus one would expect C odd operators to have an odd number of gauge fields. In a non-abelian case no operator with a single power exists, and hence the lowest non-trivial order is cubic.

Indeed, apart from renormalisation effects, the fermionic contribution (with N_f flavours) to the effective action of the bosonic Matsubara zero modes is

$$\frac{\delta\mathcal{L}_E^f}{N_f} = -ig\frac{\mu}{3}\Big(T^2+\frac{\mu^2}{\pi^2}\Big)\mathrm{Tr}\,A_0+\frac{g^2}{2}\Big(\frac{T^2}{3}+\frac{\mu^2}{\pi^2}\Big)\mathrm{Tr}\,A_0^2+i\mu\frac{g^3}{3\pi^2}\mathrm{Tr}\,A_0^3-\frac{g^4}{12\pi^2}\mathrm{Tr}\,A_0^4\,. \quad (3.2)$$

The quadratic and quartic terms here conserve C. The linear term breaks C, but is only relevant for an abelian theory where $\mathrm{Tr}\,A_0 = A_0$; it has been discussed, e.g., in [21]. The cubic term, the first C breaking operator in QCD, has been discussed in [6]. There are no higher order operators involving A_0 only [6, 22]. There are other higher order C breaking operators, though, such as

$$\delta\mathcal{L}_E^f = -i\mu N_f\frac{g^3}{3\pi^2}\frac{7\zeta(3)}{(4\pi T)^2}\Big(1+\mathcal{O}\Big(\frac{\mu}{\pi T}\Big)^2\Big)\mathrm{Tr}\,A_0F_{ij}^2\,, \quad (3.3)$$

but their contribution is suppressed at least by $\mathcal{O}(\mathbf{p}^2/(2\pi T)^2)$, where $|\mathbf{p}| \lesssim gT$ is the dynamical scale within the effective theory. Therefore we ignore them here.

For future reference, we note that if we define $j_a^\nu = -(\delta/\delta A_\nu^a)S_M$ in the non-abelian case, and $j^\nu = -(\delta/\delta A_\nu)S_M$ in the abelian, then, after the analytic continuation discussed around eq. (2.5), eq. (3.2) corresponds to having in the static limit

$$j_\nu^a = -\delta_{\nu 0}\Big[N_f\frac{g^2}{2}\Big(\frac{T^2}{3}+\frac{\mu^2}{\pi^2}\Big)A_0^a+\mu N_f\frac{g^3}{4\pi^2}d^{abc}A_0^bA_0^c\Big]\,, \quad (3.4)$$

$$j_\nu = -\delta_{\nu 0}\Big[\mu N_f\frac{g}{3}\Big(T^2+\frac{\mu^2}{\pi^2}\Big)+N_fg^2\Big(\frac{T^2}{3}+\frac{\mu^2}{\pi^2}\Big)A_0+\mu N_f\frac{g^3}{\pi^2}A_0^2\Big]\,, \quad (3.5)$$

where we have left out the contributions from the last term in eq. (3.2), going beyond the analysis in this paper.

4. The non-static case

We now move to the non-static case, to generalise the effective action in eq. (3.2). To determine the leading bosonic C odd operator in QCD, we compute the graph in figure 1.

4.1 The 3-point function

The computation of the graph in figure 1 is a straightforward exercise. Let us express the result as a contribution to the euclidean action in momentum space. Then, for an arbitrary gauge field configuration $A_\mu(x)$, we obtain

$$\delta S_E = \frac{4}{3}g^3N_f\,\sum\!\!\!\!\!\!\!\int_{Q,R,S}\delta_{Q+R+S}\,\Gamma_{\mu\nu\sigma}(Q,R,S)\,\mathrm{Tr}\Big[A_\mu(Q)A_\nu(R)A_\sigma(S)\Big], \quad (4.1)$$

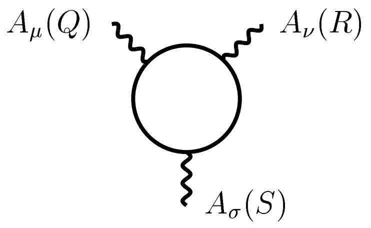

Figure 1: The Feynman graph computed in section 4.1.

where

$$\Gamma_{\mu\nu\sigma}(Q,R,S) = \sum\!\!\!\!\!\!\!\int_{P=(\omega_n^f+i\mu,\mathbf{p})} \frac{F_{\mu\nu\sigma}(p_0,\mathbf{p})}{P^2(P-Q)^2(P+S)^2}\,, \tag{4.2}$$

$$F_{\mu\nu\sigma}(p_0,\mathbf{p}) = \frac{1}{4}\mathrm{Tr}\,[\not{P}\gamma_\mu(\not{P}-\not{Q})\gamma_\nu(\not{P}+\not{S})\gamma_\sigma]. \tag{4.3}$$

To evaluate the sum over ω_n^f, we use the contour formula

$$T\sum_{n\text{ odd}} f(n\pi T+i\mu) = \sum_{\text{poles at }\mathrm{Im}\,z\neq\mu} \frac{i\,\mathrm{Res}\,f(z)}{e^{i\beta z+\beta\mu}+1}\,. \tag{4.4}$$

In addition we denote

$$\omega_P^2 = \mathbf{p}^2\,, \qquad \omega_{P-Q}^2 = (\mathbf{p}-\mathbf{q})^2\,, \qquad \omega_{P+S}^2 = (\mathbf{p}+\mathbf{s})^2\,. \tag{4.5}$$

To consider only the part odd in C, we replace in the whole expression

$$f(\mu) \to \frac{1}{2}\Big[f(\mu)-f(-\mu)\Big]\,. \tag{4.6}$$

Picking up the poles according to eq. (4.4), and using momentum conservation as well as the identity $n_{\mathrm{F}}(-\omega_P) = 1-n_{\mathrm{F}}(\omega_P)$ (cf. eq. (2.11)), the expression in eq. (4.2) becomes

$$\begin{aligned}
\Gamma_{\mu\nu\sigma} = \int_{\mathbf{p}}\Bigg\{ & \frac{N_-(\omega_P)}{4\omega_P}\Bigg[\frac{F_{\mu\nu\sigma}(-i\omega_P,\mathbf{p})}{[(q_0+i\omega_P)^2+\omega_{P-Q}^2][(s_0-i\omega_P)^2+\omega_{P+S}^2]} - (\omega_P\to-\omega_P)\Bigg] + \\
& + \frac{N_-(\omega_{P-Q})}{4\omega_{P-Q}}\Bigg[\frac{F_{\mu\nu\sigma}(q_0-i\omega_{P-Q},\mathbf{p})}{[(q_0-i\omega_{P-Q})^2+\omega_P^2][(r_0+i\omega_{P-Q})^2+\omega_{P+S}^2]} - \\
& \qquad - (\omega_{P-Q}\to-\omega_{P-Q})\Bigg] + \\
& + \frac{N_-(\omega_{P+S})}{4\omega_{P+S}}\Bigg[\frac{F_{\mu\nu\sigma}(-s_0-i\omega_{P+S},\mathbf{p})}{[(s_0+i\omega_{P+S})^2+\omega_P^2][(r_0-i\omega_{P+S})^2+\omega_{P-Q}^2]} - \\
& \qquad - (\omega_{P+S}\to-\omega_{P+S})\Bigg]\Bigg\}.
\end{aligned}$$

In fact each of these terms gives an identical contribution to eq. (4.1): changing integration variables on the second line such that $\mathbf{p} \to \mathbf{p} + \mathbf{q}$ (so that $\omega_{P-Q} \to \omega_P$), on the third line such that $\mathbf{p} \to \mathbf{p} - \mathbf{s}$ (so that $\omega_{P+S} \to \omega_P$), and in the terms obtained with ($\omega_P \to -\omega_P$) such that $\mathbf{p} \to -\mathbf{p}$, we arrive at

$$\Gamma_{\mu\nu\sigma} = -\int_{\mathbf{p}} \frac{N_-(\omega_P)}{16\omega_P^3} \Bigg[\frac{F_{\mu\nu\sigma}(-i\omega_P, \mathbf{p})}{\left(v_E \cdot Q - \frac{Q^2}{2\omega_P}\right)\left(v_E \cdot S + \frac{S^2}{2\omega_P}\right)} - \\ - \frac{F_{\mu\nu\sigma}(i\omega_P, -\mathbf{p})}{\left(v_E \cdot Q + \frac{Q^2}{2\omega_P}\right)\left(v_E \cdot S - \frac{S^2}{2\omega_P}\right)} + \\ + \Big(\text{cyclic permutations of } \mu, \nu, \sigma; Q, R, S\Big)\Bigg], \qquad (4.7)$$

where we have denoted $v_{E,\mu} = (-i, p_i/\omega_P)$. But the cyclic permutations are automatically reproduced by the trace in eq. (4.1), so we can replace them by a factor 3.

We now go to Minkowski space as discussed in the paragraph around eq. (2.5), and carry out the small effective coupling expansion. In other words, we look for the leading term in the expansion in small $Q/\omega_P, R/\omega_P, S/\omega_P$, where parametrically $Q, R, S \lesssim \max(gT, g\mu)$, while the integration variable gets its contributions from $\omega_P \sim \max(T, \mu)$ (cf. eqs. (2.12)–(2.14)). Naively, the leading term in eq. (4.7) is of order $\mathcal{O}((QS)^{-1})$. This however vanishes after cyclic permutations, due to momentum conservation (cf. eq. (4.11)). There is no term of order $\mathcal{O}(Q^{-1})$ due to the symmetries of the expression, and then the leading term is of order $\mathcal{O}(Q^0)$. In order to find it out, we need to know $F_{\mu\nu\sigma}$ to order $\mathcal{O}(Q^2)$, because it is multiplied by a term $\sim \mathcal{O}((QS)^{-1})$.

Taking traces over Dirac gamma matrices in eq. (4.3) (already transformed to Minkowski space and with indices raised), we obtain

$$F^{\mu\nu\sigma}(-i\omega_P, \mathbf{p}) = \\ = 4\omega_P^3 \Big[v^\mu v^\nu v^\sigma\Big] + \\ + 2\omega_P^2 \Big[v \cdot Q\, v^\sigma g^{\mu\nu} - v \cdot S\, v^\mu g^{\nu\sigma} - v^\mu v^\sigma Q^\nu - v^\nu v^\sigma Q^\mu + v^\mu v^\sigma S^\nu + v^\mu v^\nu S^\sigma\Big] + \\ + \omega_P \Big[-v \cdot Q\, (g^{\sigma\nu} S^\mu - g^{\mu\nu} S^\sigma - g^{\mu\sigma} S^\nu) - v \cdot S\, (g^{\mu\nu} Q^\sigma - g^{\nu\sigma} Q^\mu - g^{\mu\sigma} Q^\nu) - \\ - Q \cdot S\, (v^\nu g^{\mu\sigma} - v^\sigma g^{\mu\nu} - v^\mu g^{\nu\sigma}) - \\ - v^\sigma (Q^\mu S^\nu + S^\mu Q^\nu) - v^\mu (Q^\sigma S^\nu + Q^\nu S^\sigma) + v^\nu (Q^\sigma S^\mu - S^\sigma Q^\mu)\Big], \qquad (4.8)$$

where now $v^\mu = (1, p^i/\omega_P)$, $v \cdot v = 0$.

The terms odd in ω_P in eq. (4.8) pick up the first and third terms in the expansion of the denominators of eq. (4.7) in $Q/\omega_P, S/\omega_P$, while the term even in ω_P picks up the second term. Using in addition momentum conservation to simplify the expression,

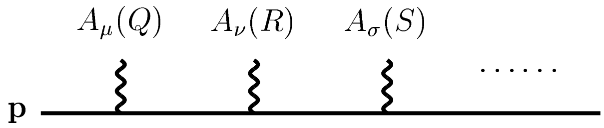

Figure 2: The Feynman graphs discussed in section 4.2.

we finally obtain

$$
\begin{aligned}
\delta S_M = & -\frac{1}{2} g^3 N_f \int_{Q,R,S} \delta_{Q+R+S} \, \mathrm{Tr} \left[A_\mu(Q) A_\nu(R) A_\sigma(S) \right] \int_{\mathbf{p}} \frac{N_-(\omega_P)}{\omega_P^2} \times \\
& \times \int_v \Bigg[\frac{v^\mu v^\nu v^\sigma}{(v\cdot Q)(v\cdot R)} \Big(\frac{(Q^2)^2}{(v\cdot Q)^2} - \frac{Q^2 R^2}{(v\cdot Q)(v\cdot R)} + \frac{(R^2)^2}{(v\cdot R)^2} \Big) + \\
& \quad + \frac{v^\mu v^\nu R^\sigma + v^\mu v^\sigma R^\nu - v^\mu v^\nu Q^\sigma - v^\nu v^\sigma Q^\mu}{(v\cdot Q)(v\cdot R)} \Big(\frac{Q^2}{v\cdot Q} - \frac{R^2}{v\cdot R} \Big) - \\
& \quad - 2 \frac{v^\sigma}{v\cdot S} \Big(g^{\mu\nu} \frac{S^2}{v\cdot S} + \frac{R^\nu S^\mu}{v\cdot R} + \frac{Q^\mu S^\nu}{v\cdot Q} \Big) + 2 \frac{S^\sigma}{v\cdot S} g^{\mu\nu} \Bigg] . \qquad (4.9)
\end{aligned}
$$

This result can still be made more transparent, however. We write the gauge field as

$$
A_\mu(Q) = \tilde{A}_\mu(Q,v) + \frac{v^\alpha Q_\mu}{v\cdot Q} A_\alpha(Q) \,, \qquad \tilde{A}_\mu(Q,v) \equiv \Big(\delta^\alpha{}_\mu - \frac{v^\alpha Q_\mu}{v\cdot Q} \Big) A_\alpha(Q) \,, \quad (4.10)
$$

such that $v\cdot\tilde{A} = 0$. It is then straightforward, employing the momentum conservation identity

$$
\frac{1}{(v\cdot Q)(v\cdot R)} + \frac{1}{(v\cdot Q)(v\cdot S)} + \frac{1}{(v\cdot S)(v\cdot R)} = 0 \,, \qquad (4.11)
$$

to show that only terms involving $\tilde{A}_\mu(Q,v)$ survive in eq. (4.9). Furthermore, since $\tilde{A}_\mu(Q,v)$ is transverse with respect to v^μ, only the last term in eq. (4.9) gives a contribution. This result is what would have been obtained if we had relied on the gauge choice $v\cdot\tilde{A} = 0$ [23, 24] to begin with. Going furthermore to x-space, we arrive at

$$
\begin{aligned}
\delta S_M &= -g^3 N_f \int_{\mathbf{p}} \frac{N_-(\omega_P)}{\omega_P^2} \int_{x,v} \mathrm{Tr} \left[\tilde{A}_\mu \tilde{A}^\mu \Big(\frac{1}{v\cdot\partial} \partial^\sigma \tilde{A}_\sigma \Big) \right] \\
&= -\frac{1}{2\pi^2} g^3 \mu N_f \int_{x,v} \mathrm{Tr} \left[\tilde{A}_\mu \tilde{A}^\mu \Big(\frac{1}{v\cdot\partial} \partial^\sigma \tilde{A}_\sigma \Big) \right] , \qquad (4.12)
\end{aligned}
$$

where the last integral was from eq. (2.14). This is the C odd part of figure 1 at leading order in the small coupling expansion, for $Q, R, S \lesssim \max(gT, g\mu) \ll \max(T, \mu)$.

4.2 Higher point functions

In a non-abelian theory, the effective action in eq. (4.12) is not explicitly gauge invariant. To obtain a gauge invariant result one has to account also for n-point functions with $n > 3$ at leading order in g. They contribute to the effective action at the same order as eq. (4.12), if we consider non-perturbative gauge field configurations such that $\partial \sim g\tilde{A} \sim g^2 T$. Fortunately such higher point functions can easily be computed, if we rely on the gauge choice $v \cdot \tilde{A} = 0$ [23, 24], as we now show.

Let us first recall the idea behind the gauge choice $v\cdot\tilde{A} = 0$. Once we have carried out the frequency sum in eq. (4.4), we are essentially evaluating the propagator of an on-shell fermion in a background gauge field (see figure 2). Because the fermion is on-shell, the result should be gauge choice independent for each $\mathbf{p}$ separately, or equivalently, for each v^μ. Thus we can choose a gauge separately for each v^μ such that $v \cdot \tilde{A} = 0$, carry out a simplified analysis in terms of $\tilde{A}_\mu$, and in the end write the result in a gauge invariant form, whereby the gauge choice can be removed.

In this gauge it is easy to see that the only higher point 1-loop graph which can contribute to the term linear in μ (such as in eq. (4.12)) is quartic in $\tilde{A}$. Indeed, after the frequency sum, the result for an n-point function is parametrically of the form (cf. eq. (4.7))

$$\delta S_E \sim \sum\!\!\!\!\!\!\!\int_Q \int_{\mathbf{p}} \frac{N(\omega_P)}{\omega_P^n} \frac{\mathrm{Tr}[\tilde{\slashed{A}}^n(Q)\slashed{P}^m\slashed{Q}^{n-m}]}{Q^{n-1}}\,, \tag{4.13}$$

where $\tilde{\slashed{A}} = \gamma_\mu \tilde{A}_\mu$, $0 \le m \le n$, and Q denotes collectively the soft external momenta. Taking the trace one obtains scalar products of $\tilde{A}$, P and the Q's. Now if $m > n/2$, there must be at least one product P^2 or $P\cdot\tilde{A}$. Both of them vanish: the first because the frequency sum puts us on the pole (cf. eq. (4.7)), and the second because of the gauge choice. Thus the leading $\mathbf{p}$-integral corresponds to $m = n/2$, and is

$$\delta S_E \sim \sum\!\!\!\!\!\!\!\int_Q \frac{\tilde{A}^n(Q)}{Q^{n/2-1}} \int_{\mathbf{p}} \frac{N(\omega_P)}{\omega_P^{n/2}}\,. \tag{4.14}$$

But this integral can lead to a linear dimensionality, i.e. a term proportional to μ, only for $n \le 4$.

We have thus computed the graph for $n = 4$,

$$\begin{aligned}\delta S_E \propto g^4 N_f &\sum\!\!\!\!\!\!\!\int_{Q,R,S,T} \delta_{Q+R+S+T} \sum\!\!\!\!\!\!\!\int_{P=(\omega_n^f+i\mu,\mathbf{p})} \\ &\times \frac{\mathrm{Tr}\,[\slashed{P}\tilde{\slashed{A}}(Q)\,(\slashed{P}-\slashed{Q})\tilde{\slashed{A}}(R)\,(\slashed{P}-\slashed{Q}-\slashed{R})\tilde{\slashed{A}}(S)\,(\slashed{P}+\slashed{T})\tilde{\slashed{A}}(T)]}{P^2(P-Q)^2(P-Q-R)^2(P+T)^2}.\end{aligned} \tag{4.15}$$

Taking again into account that $P^2 = 0$ after the frequency sum (cf. eq. (4.7)), that because of the gauge choice, $\slashed{P}$ anticommutes with $\tilde{\slashed{A}}$, and that we need the highest power of P possible in the numerator, the relevant part of the trace is

$$\mathrm{Tr}\,[\slashed{P}\tilde{\slashed{A}}(Q)\slashed{Q}\tilde{\slashed{A}}(R)\slashed{P}\tilde{\slashed{A}}(S)\slashed{T}\tilde{\slashed{A}}(T)] = 2(P\cdot Q)(P\cdot T)\mathrm{Tr}\,[\tilde{\slashed{A}}(Q)\tilde{\slashed{A}}(R)\tilde{\slashed{A}}(S)\tilde{\slashed{A}}(T)]\,. \tag{4.16}$$

But then the integral remaining is (written now in Minkowski space)

$$\delta S_M \sim g^4 N_f \int_{Q,R,S,T} \delta_{Q+R+S+T} \int_{\mathbf{p}} \frac{N_-(\omega_P)}{\omega_P^2} \int_v \frac{\mathrm{Tr}\,[\tilde{\not{A}}(Q)\tilde{\not{A}}(R)\tilde{\not{A}}(S)\tilde{\not{A}}(T)]}{v\cdot(Q+R)}\,. \qquad (4.17)$$

This however vanishes, as can be seen using momentum conservation and cyclic permutations of the trace. We thus find that the correct result for the term proportional to μ, including all 1-loop corrections, is already given by eq. (4.12).

The specific gauge choice made can now be relaxed by writing

$$\tilde{A}_\mu = \frac{1}{v\cdot\mathcal{D}} v^\alpha F_{\alpha\mu}\,, \qquad \frac{1}{v\cdot\partial}\partial^\sigma \tilde{A}_\sigma = \frac{1}{v\cdot\mathcal{D}}\mathcal{D}^\sigma \frac{1}{v\cdot\mathcal{D}} v^\gamma F_{\gamma\sigma}\,, \qquad (4.18)$$

where $\mathcal{D}$ is the covariant derivative in the adjoint representation. The right hand sides of eq. (4.18) are gauge covariant, and taking the trace in eq. (4.12) makes their product gauge invariant. Thus in an arbitrary gauge we get

$$\delta S_M = -\frac{1}{2\pi^2} g^3 \mu N_f \int_{x,v} \mathrm{Tr}\left(\frac{1}{v\cdot\mathcal{D}} v^\alpha F_{\alpha\mu}\right)\left(\frac{1}{v\cdot\mathcal{D}} v^\beta F_\beta{}^\mu\right)\left(\frac{1}{v\cdot\mathcal{D}}\mathcal{D}^\sigma\frac{1}{v\cdot\mathcal{D}} v^\gamma F_{\gamma\sigma}\right). \qquad (4.19)$$

This is our final result for the leading bosonic C odd operator, to be added to the usual Hard Thermal Loop action, given in [25].

We have not carried out explicitly the angular integral $\int_v$ in eq. (4.19). Various such integrals are discussed in [26]. The structure of these integrals is rather non-trivial and it is, for instance, not at all obvious that eq. (4.19) reduces to the third term in eq. (3.2) in the static limit, without carrying out the integration explicitly. We shall return to this issue in the next section, where we find a much simpler way of checking that the correct static limit is reproduced.

5. Classical kinetic theory

In section 4 we computed the leading bosonic C odd operator in the Hard Thermal Loop action, eq. (4.19). The form of this operator is however not particularly useful for any practical applications. Let us therefore show that the same physics is contained in a much simpler description, that of classical kinetic theory [27] (for a review, see [28]). For notational simplicity we shall work within an abelian theory for the moment, and return to the non-abelian case only in section 5.5.

5.1 Setup

We follow here the pedagogic presentation in Ref. [7] (based on the original work in [29, 30, 31]). Let us start with classical electrodynamics, and define

$$p^\alpha = \frac{dx^\alpha}{dt}\,, \qquad \frac{dp^\alpha}{dt} = -gF^\alpha{}_\beta p^\beta\,. \qquad (5.1)$$

Then the collisionless Boltzmann equation for hard particles in a gauge field background becomes

$$\frac{df(x,p)}{dt} = p^\alpha \Big(\frac{\partial f}{\partial x^\alpha} + gF_\alpha{}^\beta \frac{\partial f}{\partial p^\beta} \Big) = 0\,. \tag{5.2}$$

Let us note that we may in general assume that

$$f(x,p) = \delta_+(p^2)\hat{f}(x,p)\,, \tag{5.3}$$

where $\delta_+(p^2)$ is defined as in eq. (2.9), since this form is conserved by eq. (5.2), due to

$$p^\alpha F_\alpha{}^\beta \frac{\partial}{\partial p^\beta}\delta(p^2) = 2p^\alpha p^\beta F_{\alpha\beta}\delta'(p^2) = 0\,. \tag{5.4}$$

The derivate of $\theta(p_0)$ included in $\delta_+(p^2)$ can be safely ignored as well, since it would contribute only at the point $p_0 = \mathbf{p} = 0$, and has no effect after integration over p.

We formally solve eq. (5.2) in powers of $gF_{\mu\nu}$: $f = f_0 + f_1 + f_2 + \cdots$. This leads to the recursion relation

$$p\cdot\partial f_{n+1}(x,p) = -gp^\alpha F_{\alpha\beta}(x)\frac{\partial f_n(x,p)}{\partial p_\beta}\,. \tag{5.5}$$

The zeroth order gives

$$p\cdot\partial f_0(x,p) = 0\,. \tag{5.6}$$

We take as a solution a space-time independent function depending, in view of eq. (5.3), non-trivially only on p_0, parametrized by T,μ, and applying separately to all particle species i:

$$f_0^{(i)} = \delta_+(p^2)\,\hat{f}_0^{(i)}(p_0;T,\mu_i) = \delta_+(p^2)\,n_\mathrm{F}(p_0-\mu_i)\,, \tag{5.7}$$

where $n_\mathrm{F}(p_0)$ is in eq. (2.11). Furthermore, antiparticles are always assumed to come with the opposite signs of g and μ than particles. Thus, a single Dirac fermion contributes two degrees of freedom with $+g,+\mu$, two with $-g,-\mu$.

In addition to these equations, we need the definition of the current induced by the hard particles:

$$j^\mu(x) = -\sum_i g_i \int_P p^\mu f^{(i)}(x,p)\,. \tag{5.8}$$

The equations of motion are

$$S_M^\mathrm{free} = -\int_x \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \frac{\delta}{\delta A_\mu}S_M^\mathrm{free} = \partial_\nu F^{\nu\mu} = j^\mu\,. \tag{5.9}$$

The expression for j^μ in terms of the background gauge field thus implies a non-local effective action $S_M = S_M^\mathrm{free} + \delta S_M$ for the gauge fields only, where δS_M is to be determined from

$$\frac{\delta}{\delta A_\mu}\delta S_M = -j^\mu\,. \tag{5.10}$$

5.2 Linear term

Let us now work out explicit expressions. We start by considering $f = f_0$. Summing over N_f flavours, each with two degrees of freedom with $+g, +\mu$ and two with $-g, -\mu$, we obtain from eqs. (5.7), (5.8),

$$j^\mu = -2gN_f \int_P \delta_+(p^2)\, p^\mu N_-(p_0) = -\delta^{\mu 0}\, gN_f \frac{\mu}{3}\left(T^2 + \frac{\mu^2}{\pi^2}\right), \qquad (5.11)$$

where we used eqs. (2.10), (2.11), (2.12). Eq. (5.10) is then trivially solved. The result,

$$\delta S_M = -\int_x j^0 A_0\,, \qquad (5.12)$$

reproduces the first term in eq. (3.2) after analytic continuation.

5.3 Quadratic term

For the next term the considerations are well-known, but for completeness we briefly present them here, too. We need

$$f_1 = -g\frac{1}{p\cdot\partial}p^\alpha F_{\alpha\beta}\frac{\partial f_0}{\partial p_\beta}\,. \qquad (5.13)$$

Inserting into eq. (5.8),

$$j^\mu(x) = \sum_i g_i^2 \int_P \frac{\partial f_0^{(i)}}{\partial p_\beta}\frac{p^\mu p^\alpha}{p\cdot\partial}F_{\alpha\beta}(x)\,. \qquad (5.14)$$

As described in [7], this corresponds to the usual Hard Thermal Loop action [25]

$$\delta S_M = -g^2 N_f \int_{\mathbf{p}} \frac{N_+(\omega_P)}{\omega_P}\int_{x,v}(v^\mu F_{\mu\rho})\frac{1}{(v\cdot\partial)^2}(v^\nu F_\nu{}^\rho). \qquad (5.15)$$

To check this result at the static limit, it is actually convenient to start from the form in eq. (5.14) rather than that in eq. (5.15). We shall discuss this is detail for the non-abelian case in section 5.5, and following that argument, it is easy to see that the second term in eq. (3.5) immediately follows from eq. (5.14).

5.4 Cubic term

We then proceed to the next order in $gF_{\mu\nu}$. We are not aware of previous analyses going beyond eq. (5.15) in this way.

To proceed, it is useful to Fourier transform with respect to x:

$$f(x,p) = \int_Q e^{iQ\cdot x} f(Q,p)\,, \qquad f(Q,p) = \int_x e^{-iQ\cdot x} f(x,p)\,. \qquad (5.16)$$

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Solving for f_2 from eq. (5.5) and then inserting into eq. (5.8), we obtain

$$\begin{aligned} j^\mu(Q) &= -\sum_i g_i^3 \int_{R,S}\int_P \delta_{Q-R-S}\,\frac{p^\mu}{ip\cdot Q} p^\alpha F_{\alpha\beta}(R)\frac{\partial}{\partial p_\beta}\Big[\frac{1}{ip\cdot S}p^\delta F_{\delta\gamma}(S)\frac{\partial f_0^{(i)}}{\partial p_\gamma}\Big] \\ &= \sum_i g_i^3 \int_{R,S}\int_P \delta_{Q-R-S}\,\frac{\partial f_0^{(i)}}{\partial p_\gamma}\frac{\partial}{\partial p_\beta}\Big(\frac{p^\mu}{ip\cdot Q}\Big)p^\alpha F_{\alpha\beta}(R)\frac{1}{ip\cdot S}p^\delta F_{\delta\gamma}(S) \\ &= -\sum_i g_i^3 \int_{R,S}\int_P \delta_{Q-R-S}\, f_0^{(i)}\frac{\partial}{\partial p_\gamma}\Big[\frac{\partial}{\partial p_\beta}\Big(\frac{p^\mu}{p\cdot Q}\Big)(p\cdot R)\tilde{A}_\beta(R)\tilde{A}_\gamma(S)\Big], \end{aligned} \tag{5.17}$$

where we carried out two partial integrations and wrote (cf. eq. (4.18))

$$p^\alpha F_{\alpha\beta}(R) = i(p\cdot R)\tilde{A}_\beta(R) = i(p\cdot R)^2\frac{\partial}{\partial p^\beta}\Big(\frac{p^\nu}{p\cdot R}\Big)A_\nu(R)\,. \tag{5.18}$$

We can now take the partial derivatives $\partial/\partial p_\gamma, \partial/\partial p_\beta$, using

$$\frac{\partial \tilde{A}_\mu(Q)}{\partial p^\sigma} = -\frac{Q_\mu}{p\cdot Q}\tilde{A}_\sigma(Q)\,, \tag{5.19}$$

following from eq. (5.18) (or eq. (4.10)). By a change of summation and integration variables we can also symmetrise the expression with respect to $(R\leftrightarrow S, \beta\leftrightarrow\gamma)$, since $\tilde{A}_\beta, \tilde{A}_\gamma$ commute. Using in addition that

$$\sum_i g_i^3 f_0^{(i)} = 2g^3 N_f\,\delta_+(p^2)\,N_-(p_0), \tag{5.20}$$

we arrive at

$$\begin{aligned} j^\mu(Q) = g^3 N_f \int_{R,S}\int_P &\delta_{Q-R-S}\;\delta_+(p^2)\,N_-(p_0)\,\times \\ &\times\Big[Q^\alpha g^{\beta\gamma} + R^\beta g^{\gamma\alpha}\frac{p\cdot Q}{p\cdot R} + S^\gamma g^{\alpha\beta}\frac{p\cdot Q}{p\cdot S}\Big]\frac{\partial}{\partial p^\alpha}\Big(\frac{p^\mu}{p\cdot Q}\Big)\tilde{A}_\beta(R)\tilde{A}_\gamma(S)\,. \end{aligned} \tag{5.21}$$

Let us now show that the same expression is obtained by computing

$$j^\mu(Q) = \int_x e^{-iQ\cdot x} j^\mu(x) = -\int_x e^{-iQ\cdot x}\frac{\delta}{\delta A_\mu(x)}\delta S_M = -\int_{Q'}\delta_{Q+Q'}\frac{\delta}{\delta A_\mu(Q')}\delta S_M\,, \tag{5.22}$$

where we used $\delta A^\nu(Q')/\delta A_\mu(x) = g^{\mu\nu}e^{-iQ'\cdot x}$, and we expect (cf. eq. (4.12))

$$\delta S_M = -g^3 N_f \int_{Q,R,S}\int_P \delta_{Q+R+S}\,\delta_+(p^2)\,N_-(p_0)\tilde{A}_\alpha(R)\tilde{A}^\alpha(S)\frac{Q^\gamma}{p\cdot Q}\tilde{A}_\gamma(Q)\,. \tag{5.23}$$

Indeed, taking derivatives using the right-most side of eq. (5.18) and writing the result again in terms of $\tilde{A}_\beta, \tilde{A}_\gamma$, we immediately recover eq. (5.21). Thus the C odd operator in eq. (4.19) is reproduced by classical kinetic theory, at least in the abelian case.

It is interesting to note that to reproduce the action in eq. (4.19), we had to carry out two partial integrations in eq. (5.17). In contrast, the static limit, the last term in eq. (3.5), is much more easily obtained from the first row of eq. (5.17). We again postpone the discussion on this to the non-abelian case in the next section.

5.5 Kinetic equations for the non-abelian case

To complete the discussion in the previous sections, we review here briefly the form of the kinetic equations in the non-abelian case, and show that they reproduce the static limit discussed in section 3. Because of a considerable proliferation of formulae, without any additional physical insight involved as far as we can judge, we do not here present a complete non-static analysis, though, as we did for the abelian case.

The simplest way to display the non-abelian kinetic equations is to follow eq. (5.2), but replace f by an $N_c \times N_c$ matrix. The collisionless QCD Boltzmann equation (for finer details of terminology, see [27, 28]) for each single fundamentally charged fermionic degree of freedom, and the corresponding gauge current induced, can be written as

$$\Big[p \cdot D, f\Big] + \frac{g}{2}\Big\{p^\mu F_{\mu\nu}, \frac{\partial f}{\partial p_\nu}\Big\} = 0\,, \tag{5.24}$$

$$j^a_\mu = -g \int_P p_\mu \mathrm{Tr}\Big[T^a f\Big]\,. \tag{5.25}$$

To express this in a more familiar way, we may write the matrix f in terms of a singlet distribution function $\bar{f}$ and an adjoint (or octet) distribution function f^a:

$$f(x,p) = \frac{1}{N_c}\bar{f}(x,p) + 2T^a f^a(x,p)\,. \tag{5.26}$$

Projecting then eq. (5.24) with $\mathrm{Tr}\,[\ldots]$ and $\mathrm{Tr}\,[T^a \ldots]$, the equations obtain their usual forms (with our sign conventions) [27, 28],

$$\begin{aligned} & p \cdot \partial \bar{f} + g p^\mu F^a_{\mu\nu} \frac{\partial f^a}{\partial p_\nu} = 0\,, \\ & (p \cdot \mathcal{D})^{ab} f^b + \frac{g}{2} d^{abc} p^\mu F^b_{\mu\nu} \frac{\partial f^c}{\partial p_\nu} + \frac{g}{2N_c} p^\mu F^a_{\mu\nu} \frac{\partial \bar{f}}{\partial p_\nu} = 0\,, \\ & j^a_\mu = -\sum_i g_i \int_P p_\mu f^{a(i)}\,. \end{aligned} \tag{5.27}$$

When computing the current, each quark flavour now comes with N_c colours, in addition to two spin degrees of freedom, both for particles and for anti-particles.

The equations can again be solved iteratively in $gF_{\mu\nu}$: $f = f_0 + f_1 + f_2 + \cdots$. At the zeroth order,

$$\begin{aligned} \bar{f}^{(i)}_0 &= \delta_+(p^2)\, n_\mathrm{F}(p_0 - \mu_i)\,, \\ f^{a(i)}_0 &= 0\,. \end{aligned} \tag{5.28}$$

Iterating this, we obtain at the first and second order,

$$\bar{f}_1^{(i)} = 0\,, \tag{5.29}$$

$$(p\cdot\mathcal{D})^{ab} f_1^{b(i)} = -\frac{g_i}{2N_c} p^\mu F^a_{\mu 0}\, \delta_+(p^2)\, n'_{\rm F}(p_0-\mu_i)\,, \tag{5.30}$$

$$p\cdot\partial \bar{f}_2^{(i)} = -g_i p^\mu F^a_{\mu\nu} \frac{\partial f_1^{a(i)}}{\partial p_\nu}\,, \tag{5.31}$$

$$(p\cdot\mathcal{D})^{ab} f_2^{b(i)} = -\frac{g_i}{2} d^{abc} p^\mu F^b_{\mu\nu} \frac{\partial f_1^{c(i)}}{\partial p_\nu}\,. \tag{5.32}$$

In order to argue that these equations contain the same physics as eq. (4.19), we shall complement the full proof for the abelian case in section 5.4, by showing that the full non-abelian static limit of eq. (3.4) is also reproduced. To demonstrate the latter point, we start from the first order in $gF_{\mu\nu}$. In eq. (5.30) we can write

$$p^\mu F^a_{\mu 0} = (p\cdot\mathcal{D}A_0)^a - \partial_0(p^\mu A^a_\mu) = (p\cdot\mathcal{D})^{ab} A^b_0\,, \tag{5.33}$$

so that

$$f_1^{a(i)} = -\frac{g_i}{2N_c} A^a_0\, \delta_+(p^2)\, n'_{\rm F}(p_0-\mu_i)\,. \tag{5.34}$$

Inserting into eq. (5.27), we obtain

$$\begin{aligned} j^a_\mu = -\sum_i g_i \int_P p_\mu f_1^{a(i)} &= g^2 N_f A^a_0 \int_P p_\mu\, \delta_+(p^2)\, N'_+(p_0) \\ &= \delta_{\mu 0}\, g^2 N_f A^a_0 \int_{\mathbf{p}} N'_+(\omega_P)\,. \end{aligned} \tag{5.35}$$

After use of eq. (2.13), this agrees with eq. (3.4).

The equation for f_2^a, on the other hand, becomes

$$\begin{aligned} (p\cdot\mathcal{D})^{ab} f_2^{b(i)} &= \frac{g_i^2}{4N_c} d^{abc} p^\mu F^b_{\mu 0} A^c_0\, \delta_+(p^2)\, n''_{\rm F}(p_0-\mu_i) \\ &= \frac{g_i^2}{4N_c} d^{abc} (p\cdot\mathcal{D}A_0)^b A^c_0\, \delta_+(p^2)\, n''_{\rm F}(p_0-\mu_i)\,. \end{aligned} \tag{5.36}$$

Here

$$d^{abc}(p\cdot\mathcal{D}A_0)^b A^c_0 = \frac{1}{2}(p\cdot\mathcal{D})^{ab}(d^{bcd}A^c_0 A^d_0)\,, \tag{5.37}$$

so that

$$f_2^{a(i)} = \frac{g_i^2}{8N_c} d^{abc} A^b_0 A^c_0\; \delta_+(p^2)\, n''_{\rm F}(p_0-\mu_i)\,. \tag{5.38}$$

Carrying out the sum over i in eq. (5.27), we finally get

$$\begin{aligned} j^a_\mu &= -\frac{1}{4} g^3 N_f d^{abc} A^b_0 A^c_0 \int_P p_\mu\, \delta_+(p^2)\, N''_-(p_0) \\ &= -\delta_{\mu 0}\, \frac{1}{4} g^3 N_f d^{abc} A^b_0 A^c_0 \int_{\mathbf{p}} N''_-(\omega_P)\,. \end{aligned} \tag{5.39}$$

Using eq. (2.14), this agrees with eq. (3.4).

5.6 Possible simplifications of the classical description?

We have argued that classical kinetic theory should reproduce the results of the Hard Thermal Loop effective theory even for C odd observables, thus correctly representing the infrared physics of the system. It has however some merits over the Hard Thermal Loop action: it is local, directly minkowskian and, involving only the solution of classical equations of motion, can be implemented numerically without the need for analytic continuation. Moreover, judging from non-perturbative studies of dimensional reduction, such numerical results should be at least qualitatively reliable for $T \sim (2T_c, \infty)$.

In full generality, the classical distribution functions $f(x, p)$ depend on (4+4) coordinates. If one wants to proceed towards numerical implementation, it is best to reduce the dimensionality of the phase space. As we have discussed, $f(x, p) = \delta_+(p^2)\hat{f}(x, p)$, and the dependence can thus be reduced to, say, the spatial components $\mathbf{p}$. This (4+3) dimensional problem can already be managed on the lattice [32]. However, for the usual C even case one can carry out a further simplification, by writing in eq. (5.30)

$$f_1^{a(i)} = -\frac{g_i}{2N_c}\,\delta_+(p^2)\,n'_{\mathrm{F}}(p_0 - \mu_i)\,W_1^a(x, \mathbf{v})\,, \tag{5.40}$$

and performing the sum over i and the integral over p_0 in eq. (5.27), reducing thus the dependence only to angular variables and a total of (4+2) dimensions. Further simplifications in the angular variables (like an expansion in spherical harmonics [33, 34]) may also be possible.

It is then natural to ask whether a similar simplification could be made in the presence of $f_2^{a(i)}$. In order to have a proper statistical weighting of the initial conditions for the time evolution, one should also work out a hamiltonian formulation in terms of the gauge fields A_i^a, $E_i^a \equiv F_{0i}^a$ and W_1^a, W_2^a, an issue which is not altogether trivial [35, 36]. We have not carried out these constructions, but are not aware of any *a priori* fundamental problems in doing so.

6. Conclusions

In this paper we have addressed the computation of real time observables in QCD at finite temperatures and densities.

Most of the physical observables one can think of are not computable in perturbation theory, due to infrared problems. Real time observables are not well suited for direct 4d lattice simulations, either. However, it is possible to use perturbation theory to construct an effective description of the infrared dynamics. If simple enough, this could then be studied by non-perturbative means.

We have here completed the bosonic sector of the Hard Thermal Loop effective theory for real time observables, by computing the leading C odd operator induced by a finite density. This operator is of a relatively simple form (eq. (4.19)) and, as

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we have argued, is reproduced by classical kinetic theory, whose equations of motion are local (section 5.5).

The classical kinetic theory can then be studied on the lattice, as has been demonstrated in the case of the sphaleron rate in the electroweak theory [32, 34], as well as in the case of the defect formation rate in scalar electrodynamics [33]. Strictly speaking there are still problems in finding a formulation which has a well-defined continuum limit [37], but these may not be important for practical applications, in which one may hope to find an intermediate scaling window allowing to extract physical results.

When the chemical potential is turned on, the further problem arises that the "hamiltonian" determining the thermal distribution of initial conditions is complex, resulting in a "sign problem": standard Monte Carlo methods are not efficient in sampling the configuration space, when one is close to the thermodynamic (infinite volume) limit. Numerical tests in the static case have shown, though, that in practice one can reach large enough volumes before the sign problem starts to reduce the signal-to-noise ratio in any significant way [6, 38].

Thus, it seems that a numerical determination of many non-perturbative real time quark-gluon plasma observables may become feasible, using classical kinetic theory.

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